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Quantum spin coverings and statistics

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Abstract

$SL_q(2)$ at primitive odd roots of unity $q^\ell = 1$ is studied as a quantum cover of the complex rotation group $SO(3,\mathbb{C})$, in terms of the associated Hopf algebras of (quantum) polynomial functions. We work out the irreducible corepresentations, the decomposition of their tensor products and a coquasitriangular structure, with the associated braiding (or statistics). As an example, the case $\ell = 3$ is discussed in detail.

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1. Introduction

As is well known, the twofold spin covering $\mathbb{Z}_2 \to Spin(2) \to SO(2)$ is not universal and there are other (nontrivial) coverings with kernel $\mathbb{Z}_n$ or $\mathbb{Z} = \pi_1(SO(2))$ (for the universal one). They are responsible for features such as the fractional or continuous spin and the associated anyonic statistics (see, e.g., [15, 30]). In dimension 3 (or greater) $\pi_1(SO(3)) = \mathbb{Z}_2$ and the spin covering $\mathbb{Z}_2 \to Spin(3) \to SO(3)$ is universal; hence, the only (projective) representations are of half-integer spin (in addition to those of integer spin which are *bona fide* representations of $SO(3)$). The quantum groups offer a possibility of refining this classification. There is indeed a candidate for such a cover, the quantum group $SL_q(2)$ at the roots of unity [5]. The case of the third root $q = e^{\frac{2\pi i}{3}}$ has been worked out quite extensively mainly in terms of the quantum universal enveloping algebra $U_q(sl(2))$ (cf [6, 18] and references therein) with the perspective to link to the Connes' algebra for the standard model. Also the Hopf algebra $A(SL_q(2))$ of 'polynomials on $SL_q(2)$' at odd roots of unity $q^\ell = 1$ has been studied. We adopt this 'quantum function' point of view since it is better suited to capture topological properties such as quantum (finite) covers, which have to be introduced by hand when working in the universal enveloping algebra language. In section 2, after recalling the essentials on $A(SL_q(2))$, we present the coverings of $Spin(3,\mathbb{C})$ and $SO(3,\mathbb{C})$.

The main point we are interested in this paper is to study further this new quantum symmetry and the braiding (or statistics) of its *corepresentations*. The labelling of irreducible corepresentations refines the notion of *spin*. On the dual level, the irreducible representations of $U_q(sl(2))$ at roots of unity are well known and the peculiar decomposition of their tensor products has been studied, e.g., in [1, 3, 19, 27]. They are not of immediate use for us since the dual pairing between $A(SL_q(2))$ and $U_q(sl(2))$ degenerates at roots of unity. Actually it descends to a (nondegenerate) duality between certain finite-dimensional quotients $A(F)$ and $\bar{U}_q$ of these Hopf algebras. The irreducible representations of $\bar{U}_q$ are also known [17] and by the duality they correspond to irreducible corepresentations of $A(F)$, but this is not tantamount to those of $A(SL_q(2))$. There exists another Hopf algebra of divided powers [20, 21], or a version of it due to [12], which is dual to $A(SL_q(2))$ [10, 11]. Thus the irreducible corepresentations of $A(SL_q(2))$ should be in correspondence with the irreducible representations of divided powers Hopf algebra, which are known. This reasoning requires however some algebraic subtleties, which we would like to avoid. For the sake of the physicists' community, in section 3 we provide a direct computational proof of the irreducibility. The first nontrivial case $\ell = 3$ is discussed in subsection 3.1, where we study the decomposition of tensor products of the irreducible corepresentations of $A(SL_q(2))$ and the question if it is possible to build the fundamental (spin 1/2) corepresentation out of three 'fractional' corepresentations.

As far as the braiding is concerned, it is associated with a coquasitriangular structure. In section 4 we show that the coquasitriangular structure obtained from the standard universal R matrix associated with $U_q(sl(2))$ leads to a fairly exotic braiding, which however is consistent with the Bose–Fermi statistics of the usual (half)integer spin corepresentations. As the first nontrivial example, the braiding in the case $\ell = 3$ is discussed in detail in subsection 4.1. Section 5 contains final remarks and conclusions.

2. Preliminaries

We start by recalling the basic definitions and our notational conventions.

2.1. The quantum group $SL_q(2)$

Recall that for $\mathbb{C} \ni q \neq 0$, $A(SL_q(2))$ is the unital free algebra generated by a, b, c, d over $\mathbb{C}$ modulo the ideal generated by the commutation relations

$$\begin{aligned} &ab = qba \qquad &&ac = qca \qquad &&bd = qdb \\ &bc = cb \qquad &&cd = qdc \qquad &&ad - da = (q - q^{-1})bc \end{aligned}$$

and by the q-determinant relation $ad - qbc = 1$. The Hopf algebra structure is given by the following comultiplication Δ, counit ε, and antipode S defined on the generators (arranged as a 2×2 matrix) by

$$\begin{aligned} &\Delta \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ &\varepsilon \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad S \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & -q^{-1}b \\ -qc & a \end{pmatrix} \end{aligned} \tag{1}$$

where on the rhs of the first equation the 'line by columns' tensor product is understood. As a complex vector space, $A(SL_q(2))$ has a basis $a^i b^j c^k$ with $i, j, k \in \mathbb{N}$ and $b^i c^j d^k$ with $i, j \in \mathbb{N}, k \in \mathbb{Z}_+$. (We denote $\mathbb{Z}_+ = \{1, 2, 3, \ldots\}$ and $\mathbb{N} = \{0, 1, 2, \ldots\}$.)

2.2. *Finite quantum subgroups F and $\hat{F}$*

From now on, unless stated differently, we set the parameter q to be a (primitive) ℓth root of unity $\lambda = e^{\frac{2\pi i}{\ell}}$, for odd $\ell \geqslant 3$. We introduce two finite-dimensional Hopf algebras of 'functions on finite quantum subgroups' F and $\hat{F}$ of $SL_q(2)$, needed in the following.

The Hopf algebra $A(F)$ is defined as the quotient Hopf algebra of $A(SL_q(2))$ modulo the ideal generated by the relations

$$a^\ell = 1 = d^\ell \qquad b^\ell = 0 = c^\ell. \tag{2}$$

Let π_F denote the canonical projection, and $\tilde{t} := \pi_F(t)$.

We give now some information on F (see [7, 8] for the case $\ell = 3$).

Proposition 2.1. *$A(F)$ satisfies the following properties:*

(i) as a complex vector space $A(F)$ is ℓ^3-dimensional and its basis can be chosen as $\tilde{a}^p\tilde{b}^r\tilde{c}^s$, where $p, r, s \in \{0, 1, \ldots, \ell - 1\}$.

(ii) $A(F)$ has a faithful representation ϱ

$$\varrho(\tilde{a}) = \mathbf{J} \otimes \mathbf{1}_\ell \otimes \mathbf{1}_\ell \qquad \varrho(\tilde{b}) = \mathbf{Q} \otimes \mathbf{N} \otimes \mathbf{1}_\ell \qquad \varrho(\tilde{c}) = \mathbf{Q} \otimes \mathbf{1}_\ell \otimes \mathbf{N} \tag{3}$$

where for $i, j \in \{1, 2, \ldots, \ell\}$

$$\begin{gathered} \mathbf{J}_{i,j} = \begin{cases} 1 & \text{if} \quad i = j+1 \bmod \ell \\ 0 & \text{otherwise} \end{cases} \\ \mathbf{Q}_{i,j} = \begin{cases} q^{-i} & \text{if} \quad i = j \\ 0 & \text{otherwise} \end{cases} \qquad \mathbf{N}_{i,j} = \begin{cases} 1 & \text{if} \quad i = j+1 \\ 0 & \text{otherwise.} \end{cases} \end{gathered} \tag{4}$$

(iii) F has the 'reduced' quantum plane as a quantum homogeneous space, i.e. the algebra generated by x and y modulo the ideal generated by the relations $xy = qyx$, $x^\ell = 1$ and $y^\ell = 1$ (isomorphic to $\mathrm{Mat}(\ell, \mathbb{C})$*) is an $A(F)$-comodule algebra.*

(iv) F has a classical subgroup, defined as the group of characters of $A(F)$, which is easily seen to be isomorphic to $\mathbb{Z}_\ell$. Namely, for $i \in \{1, 2, \ldots, \ell\}$ we have χ_i defined by their action on the generators as $\chi_i(\tilde{a}) = q^i$, $\chi_i(\tilde{b}) = 0$, $\chi_i(\tilde{c}) = 0$ and $\chi_i(\tilde{d}) = q^{-i}$. The Hopf algebra $A(\mathbb{Z}_\ell)$ appears as a quotient of $A(F)$ by the ideal generated by $\tilde{b}, \tilde{c}$ (which is also the intersection of the kernels of the characters).

Quite similarly, we define $A(\hat{F})$ as the $2\ell^3$-dimensional quotient of $A(SL_q(2))$ modulo the relations

$$a^{2\ell} = 1 = d^{2\ell} \qquad b^\ell = 0 = c^\ell. \tag{5}$$

Note that the classical subgroup of $\hat{F}$ is the cyclic group $\mathbb{Z}_{2\ell}$. This group 'combines' the cyclic subgroup $\mathbb{Z}_\ell$ of F with $\mathbb{Z}_2$ appearing in the classical spin cover. In fact, one has the exact sequence of groups

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathbb{Z}_{2\ell} \longrightarrow \mathbb{Z}_\ell \longrightarrow 0 \tag{6}$$

which extends $\mathbb{Z}_\ell$ by the kernel $\mathbb{Z}_2$ of the classical spin cover. Note that this extension is a direct product of groups for odd ℓ, while it is not even a semidirect product for even ℓ.

2.3. Quantum group covering of $SL(2)$

The Hopf subalgebra of $A(SL_q(2))$ generated by the ℓth powers

$$\alpha = a^l \qquad \beta = b^l \qquad \gamma = c^l \qquad \delta = d^l$$

is isomorphic to the (commutative) Hopf algebra

$$A(SL(2)) = \mathbb{C}[\alpha, \beta, \gamma, \delta]/\langle \alpha\delta - \beta\gamma - 1\rangle$$

with the restricted coproduct, counit and coinverse. It is just the subalgebra of coinvariants of the coaction of $A(F)$ (as a quotient Hopf algebra). It is known [2] (see also [7] for the case $\ell = 3$) that

Proposition 2.2. *The sequence of algebras*

$$A(SL(2)) \longrightarrow A(SL_q(2)) \xrightarrow{\pi_F} A(F) \tag{7}$$

is

(i) a (right, faithfully flat) Hopf–Galois extension of $A(SL(2)$ by $A(F)$ (quantum principal fibre bundle),
(ii) a principal homogeneous Hopf–Galois extension (i.e. $A(F)$ is a quotient of the Hopf algebra $A(SL_q(2))$ by a Hopf ideal and π_F is the canonical surjection),
(iii) strictly exact (quantum quotient group).

(Note that (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i), see [26, 28, 29] for the relevant definitions.)

Therefore, (7) is a good candidate for a quantum covering of the spin group. To be fully worthy of this name, it would be better nontrivial. It can be seen that it is not totally trivial in the sense that $A(SL_q(2))$ is not isomorphic to $A(SL(2)) \otimes A(F)$. Another accepted notion to substitute the triviality for quantum principal bundles is however that of cleftness (or crossed products), cf, e.g., [7]. It is not yet known if (7) is cleft and it seems to be a tough problem, which is not tractable by the usual means (e.g., the theory of quantum characteristic classes). A weaker result affirms that $A(SL_q(2))$ as a module over $A(SL(2))$, which is finitely generated and projective (cf [11]), is actually free [9]. (The associated coherent sheaf of rank l^3 is free and the corresponding vector bundle F over $SL(2)$ turns out to be trivial.) Moreover a set of l^3 generators can be chosen as [9]

$$a^m b^n c^{s'} \qquad b^n c^{s''} d^r$$

with the integers m, n, r, s', s'' in the range $m \in \{1, \ldots, \ell - 1\}, n, r \in \{0, \ldots, \ell - 1\}$, $s' \in \{m, \ldots, \ell - 1\}$ and $s'' \in \{0, \ldots, \ell - r - 1\}$.

We expect that the quantum principal bundle (7) is actually noncleft and propose to employ it for a quantum spin covering in the next section.

2.4. Quantum group covering of $SO(3, \mathbb{C})$

The (complex) group $SL(2)$ is isomorphic to the spin group $Spin(3, \mathbb{C})$ and thus provides a twofold covering of the (complex) rotation group $SO(3, \mathbb{C})$. This classical spin covering can be combined with the covering (7) as follows. The Hopf algebra $A(SO(3, \mathbb{C}))$ can be identified with the subalgebra of even polynomials $A^+(SL(2))$ in $A(SL(2))$. It can be seen that $A^+(SL(2))$ coincides with the coinvariants of the coaction of Hopf algebra $A(\hat{F})$. Let $\pi_{\hat{F}}$ denote the canonical projection and $\hat{t} := \pi_{\hat{F}}(t)$. In analogy with the proof of proposition 2.2, it can be shown that

Proposition 2.3. *The sequence of Hopf algebras*

$$A(SO(3,\mathbb{C})) \longrightarrow A(SL_q(2)) \xrightarrow{\pi_{\hat{F}}} A(\hat{F}) \tag{8}$$

possesses the same nice properties (i)–(iii) as the sequence (7).

In particular, (8) is a quantum principal bundle and referring to our remarks at the end of the previous subsection, we mention that it is very likely noncleft. In fact, the relevant question about (8) is whether it is reducible to the subgroup $\mathbb{Z}_2$. We expect that it is not the case, consistently with our conjecture about the noncleftness of (7) and thus propose the following definition.

Definition. *For any odd ℓ, $\ell \geqslant 3$, with $q = \mathrm{e}^{\frac{2\pi \mathrm{i}}{\ell}}$, we call the principal fibre bundle (8) quantum spin covering of the complex rotation group.*

3. Irreducible corepresentations

There are two natural series of corepresentations of $A(SL_q(2))$. The first one comes by restricting the coproduct to $W_n = \mathbb{C}\{\alpha^n, \alpha^{n-1}\gamma, \dots, \gamma^n\}$, i.e. the span of monomials of degree n in $\alpha = a^\ell$ and $\gamma = c^\ell$. The corepresentation W_n is a 'push forward' of the usual $(n+1)$-dimensional (spin $n/2$) corepresentations of $A(SL(2))$ and thus it is obviously irreducible for all $n \in \mathbb{N}$. The second one comes by restricting the coproduct to $Y_m = \mathbb{C}\{a^m, a^{m-1}c, \dots, c^m\}$, i.e. the span of monomials of degree m in a, c. More explicitly,

$$\begin{aligned}\Delta a^{m-h}c^h &= \sum_{r=1}^{m-h}\sum_{s=1}^{h} q^{-r(h-s)}\binom{m-h}{r}_{q^{-2}}\binom{h}{s}_{q^{-2}} a^{m-h-r}b^r c^{h-s}d^s \otimes a^{m-(r+s)}c^{r+s} \\ &= \sum_{k=0}^{m}\left(\sum_{r+s=k} q^{-r(h-s)}\binom{m-h}{r}_{q^{-2}}\binom{h}{s}_{q^{-2}} a^{m-h-r}b^r c^{h-s}d^s\right)\otimes a^{m-k}c^k \end{aligned} \tag{9}$$

where

$$\binom{k}{j}_p = \frac{(k)!_p}{(k-j)!_p (j)!_p} \qquad (k)!_p = (k)_p (k-1)_p \cdots (2)_p \qquad \text{and}$$
$$(k)_p = 1 + p + \cdots + p^{k-1}.$$

It is indecomposable but not irreducible in general. We shall see that, for $m \in \{0, 1, \dots, \ell-1\}$, Y_m is indeed irreducible and we shall denote it by V_m. Also, for $m = n\ell - 1, n \in \mathbb{Z}_+$, it is irreducible but in fact equivalent to $W_{n-1} \otimes V_{\ell-1}$. More generally, the corepresentations of the form $W_n \otimes V_m$, with $n \in \mathbb{N}, m \in \{0, 1, \dots, \ell-1\}$ are all irreducible as can be inferred from [10, 11], establishing the duality with a version [12] of divided powers algebra [20, 21] of which the classification of irreducible representations is known [20]. Although straightforward, here we provide a direct computational proof.

Proposition 3.1. *Set $m = m_0 + \ell m_1$, with $0 \leqslant m_0 \leqslant \ell - 1$, $m_1 \geqslant 0$.*

(a) For $m_1 = 0$ the comodule $V_{m_0} := Y_{m_0}$ is irreducible.
(a′) For every $m_1 > 0$, the comodule $Y_{\ell-1+\ell m_1}$ (i.e. when $m_0 = \ell - 1$) is irreducible as well and it is isomorphic to $W_{m_1} \otimes V_{\ell-1}$.
(b) When $0 \leqslant m_0 \leqslant \ell - 2$ and $m_1 \geqslant 1$, $Y_{m_0+\ell m_1}$ has a (maximal) subcomodule isomorphic to $W_{m_1} \otimes V_{m_0}$. The quotient comodule is irreducible and isomorphic to $W_{m_1-1} \otimes V_{\ell-2-m_0}$.

Corollary. *The corepresentations* $W_n \otimes V_m$ *are irreducible for all* $n \in \mathbb{N}$ *and* $m \in \{0, 1, \ldots, \ell - 1\}$.

Proof of proposition 3.1. The classical argument working for $q = 1$ can be directly extended when q is considered as an indeterminate and runs as follows. Given a corepresentation ρ of a Hopf algebra A on a comodule U, let ρ_i^j be a matrix of elements of A such that

$$u_i \mapsto \rho_i^j \otimes u_j$$

with respect to a basis u_i $(i = 1, \ldots, n)$ of U. There exists a coinvariant subcomodule $U' \subset U$ (with $\dim U' = k$, say) iff up to a conjugation by an invertible matrix Z with elements in $\mathbb{C}[q, q^{-1}]$ the matrix ρ takes a lower echelon form, i.e. iff

$$\begin{pmatrix} \tau_1 & 0 \\ \tau_3 & \tau_4 \end{pmatrix} \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix} = \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix} \begin{pmatrix} \rho_1 & \rho_2 \\ \rho_3 & \rho_4 \end{pmatrix}$$

where U' is the span of the first k elements of the transformed basis and the block decomposition is given by the splitting $U = U' \oplus U/U'$. In particular, this requires that

$$\tau_1 (z_1 \quad z_2) = (z_1 \rho_1 + z_2 \rho_3 \quad z_1 \rho_2 + z_2 \rho_4).$$

Note that the $k \times n$ matrix $(\, z_1 \quad z_2 \,)$ has rank k. Let M be an invertible $k \times k$ submatrix. We can write

$$\tau_1 M = S \qquad \tau_1 M' = S'$$

where on the rhs S is the submatrix corresponding to the columns of M in $(\, z_1 \quad z_2 \,)$ and a prime denotes the submatrix with the complementary columns. Substituting, we get $k \times (n - k)$ linear relations over $\mathbb{C}[q, q^{-1}]$ among the elements ρ_i^j,

$$S' = S M^{-1} M'.$$

Now let us have a closer look at the comodules Y_m. The matrix elements of (9) are linear combinations of monomials of degree m in the generators a, b, c, d. For generic q, all $(m + 3)(m + 2)(m + 1)/6$ of them appear in the sum on the rhs of (9) and every monomial appears exactly in a single matrix element, i.e. two different matrix elements contain different monomials. Therefore, they are all linearly independent and, from the above argument, Y_m are irreducible.

Recall that when $q = \lambda$ is an ℓth root of unity, $\lambda^\ell = 1$, the subalgebra generated by $\alpha = a^\ell, \beta = b^\ell, \gamma = c^\ell, \delta = d^\ell$ is central and isomorphic to the classical Hopf algebra $A(SL(2))$. Now several q^{-2}-binomial coefficients actually vanish when evaluated at λ. A simple way to control this is to use the fact that the coproduct is an algebra homomorphism

$$\Delta a^m = \Delta \alpha^{m_1} \Delta a^{m_0}$$

where $m = m_0 + \ell m_1$. Hence

$$\sum_{r=0}^{m} \binom{m}{r}_{\lambda^{-2}} a^{m-r} b^r \otimes a^{m-r} c^r$$
$$= \sum_{i=0}^{m_1} \sum_{j=0}^{m_0} \binom{m_0}{j}_{\lambda^{-2}} \binom{m_1}{i}_1 \alpha^{m_1 - i} \beta^i a^{m_0 - j} b^j \otimes \alpha^{m_1 - i} \gamma^i a^{m_0 - j} c^j$$

giving the factorization formula (cf [20])

$$\binom{m}{r}_{\lambda^{-2}} = \binom{m_0}{r_0}_{\lambda^{-2}} \binom{m_1}{r_1} \qquad (10)$$

where $m = m_0 + \ell m_1$, $r = r_0 + \ell r_1$, $0 \leqslant m_0, r_0 \leqslant \ell - 1$, $m_1, r_1 \geqslant 0$ and the last factor on the rhs is just the ordinary binomial coefficient. In particular, all the binomial coefficients $\binom{m}{r}_{\lambda^{-2}}$ with $r_0 > m_0$ vanish.

When $0 \leqslant m = m_0 \leqslant \ell - 1$, this cannot occur and the standard argument above yields point (a). For larger m, however, the comodule Y_m is no longer irreducible. Using again the homomorphism property of the coproduct, we directly compute

$$\Delta(a^{m-h}c^h) = \Delta(\alpha^{(m-h)_1}\gamma^{h_1}) \sum_{j=0}^{(m-h)_0} \sum_{t=0}^{h_0} \lambda^{-j(h_0-t)} \binom{(m-h)_0}{j}_{\lambda^{-2}} \binom{h_0}{t}_{\lambda^{-2}} \times a^{(m-h)_0-j} b^j c^{h_0-t} d^t \otimes a^{(m-h)_0+h_0-(j+t)} c^{j+t}.$$

Note that

$$m - h = \begin{cases} \ell(m_1 - h_1) + (m_0 - h_0) & \text{if} \quad 0 \leqslant h_0 \leqslant m_0 \\ \ell(m_1 - h_1 - 1) + (\ell + m_0 - h_0) & \text{if} \quad m_0 + 1 \leqslant h_0 \leqslant \ell - 1 \end{cases}$$

and the above formula for $h_0 \leqslant m_0$ reads

$$\Delta(a^{m-h}c^h) = \Delta(\alpha^{m_1-h_1}\gamma^{h_1}) \sum_{j=0}^{m_0-h_0} \sum_{t=0}^{h_0} \lambda^{-j(h_0-t)} \binom{m_0-h_0}{j}_{\lambda^{-2}} \binom{h_0}{t}_{\lambda^{-2}} \times a^{m_0-h_0-j} b^j c^{h_0-t} d^t \otimes a^{m_0-(j+t)} c^{j+t}. \tag{11}$$

This sum contains only monomials in a, c of degree m_0. So $W_{m_1} \otimes V_{m_0}$ is an irreducible subcomodule. If $m_0 = \ell - 1$, this is isomorphic to the whole of $Y_{\ell-1+\ell m_1}$. This proves point (a′) and the first statement of point (b). To complete the proof of (b) note that for $m_0 + 1 \leqslant h_0 \leqslant \ell - 1$, we have

$$\Delta(a^{m-h}c^h) = \Delta(\alpha^{m_1-h_1-1}\gamma^{h_1}) \sum_{j=0}^{\ell+m_0-h_0} \sum_{t=0}^{h_0} \lambda^{-j(h_0-t)} \binom{\ell+m_0-h_0}{j}_{\lambda^{-2}} \binom{h_0}{t}_{\lambda^{-2}} \times a^{\ell+m_0-h_0-j} b^j c^{h_0-t} d^t \otimes a^{\ell+m_0-(j+t)} c^{j+t}.$$

Restricting the sum to $j + t \leqslant m_0$ we can factor $a^\ell = \alpha$, while restricting to $j + t \geqslant \ell$ we can factor $c^\ell = \gamma$, thus compensating the -1 occurring in the exponent of the classical part of the coproduct and leaving in these two partial sums only monomials of degree m_0 in a, c. This cannot be done in the partial sum for $m_0 + 1 \leqslant j + t \leqslant l - 1$, which gives

$$\begin{aligned} &\sum_{k=m_0}^{l-1} \sum_{s=0}^{k} \cdots a^{l+m_0-h_0-k+s} b^{k-s} c^{h_0-s} d^s \otimes a^{l+m_0-k} c^k \\ &\quad = \sum_{k'=0}^{l-m_0-2} \sum_{s}^{m_0+1+k'} \cdots a^{m_0'-h_0'-k'+s} b^{m_0+1+k'-s} c^{m_0+1+h_0'-t} d^t \otimes (ac)^{m_0+1} a^{m_0'-k'} c^{k'} \\ &\quad = \lambda^n [(bc)^{m_0+1} \otimes (ac)^{m_0+1}] \Delta a^{m_0'-h_0'} c^{h_0'} \end{aligned}$$

where $m_0' = l - m_0 - 2$, $h_0' = h_0 + m_0 + 1$. This expression contains monomials of degree larger than m_0. Thus the quotient $Y_m / W_{m_1} \otimes V_{m_0}$ is isomorphic to $W_{m_1-1} \otimes V_{\ell-2-m_0}$ which completes the proof. □

We remark that the proof for generic q is just the q-analogue of what happens in the classical case. It is enough to note that replacing the ordinary binomial coefficients by their q-analogue one gets (9) up to some nonvanishing factors. This is why the corepresentation theory for generic q is the same as the classical one. In particular, the above argument yields that the Clebsch–Gordan decomposition holds as in the classical case

$$Y_m \otimes Y_{m'} = Y_{m+m'} \oplus Y_{m+m'-2} \oplus \cdots \oplus Y_{|m-m'|}.$$

In the case of q being ℓth root of unity, it follows from proposition 3.1 that there is always at least one irreducible corepresentation of arbitrary dimension D, and there are more (up to ℓ) of them, depending on how many integers in $\{1, \ldots, \ell\}$ divide D.

The decomposition rules of the tensor products $(W_n \otimes V_m) \otimes (W_{n'} \otimes V_{m'})$ into irreducible corepresentations follow from the usual Clebsch–Gordan decomposition of $W_n \otimes W_{n'}$ and the decomposition of $V_m \otimes V_{m'}$, which obeys a more complicated pattern.

3.1. Decomposition of tensor products for $\ell = 3$

The previous discussion can be specified and simplified considerably in the simplest (nontrivial) case $\ell = 3$. We have explicitly the following matrices ρ of three corepresentations V_0, V_1 and V_2, respectively,

$$1, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a^2 & -q^2ab & b^2 \\ ac & ad + q^{-1}bc & bd \\ c^2 & -q^2cd & d^2 \end{pmatrix}$$

as well as the usual form of W_n.

According to the corollary of proposition 3.1, we see that there is one trivial (one-dimensional, irreducible) corepresentation $V_0 = W_0$. In dimension 2 there are two (inequivalent) irreducible corepresentations V_1 and W_1. In dimension 3 there are also two: V_2 and W_2. In dimension 4, Y_3 is indecomposable but not irreducible but there are two other irreducible corepresentations $W_3 \otimes V_0 = W_3$ and $W_1 \otimes V_1$. In dimension 5 there is only one irreducible corepresentation W_4 (Y_4 is indecomposable but not irreducible). In dimension 6 there are three irreducible corepresentations: W_5, $W_2 \otimes V_1$ and $W_1 \otimes V_2 = V_5$. A general pattern is that in any dimension D there is always at least one irreducible corepresentation, if either 2 or 3 divides d, there are two irreducible corepresentations and if 6 divides D, there are three irreducible corepresentations.

We now give the decomposition rules of the tensor products.

Clearly,

$$V_0 \otimes V_0 = V_0 \qquad V_0 \otimes V_1 = V_1 \qquad \text{and} \qquad V_0 \otimes V_2 = V_2.$$

Next, it can be seen that

$$V_1 \otimes V_1 = V_0 \oplus V_2 \qquad V_1 \otimes V_2 = V_1 \oslash W_1 \oslash V_1 \qquad \text{and}$$
$$V_2 \otimes V_2 = V_0 \oslash V_2 \oslash (W_1 \oplus V_1) \oslash V_0$$

where $\oslash$ in an indecomposable corepresentation indicates that the left summand is a subcomodule while the right summand is a comodule after quotienting the left one. Noting that the tensor products in the opposite order decompose equivalently, and recalling the usual decomposition $W_n \otimes W_{n'} = W_{|n-n'|} \oplus W_{|n-n'|+2} \oplus \cdots \oplus W_{|n+n'|}$, these rules permit us to find a decomposition of tensor products of any number of $V_m \otimes W_n$, with $n \in \mathbb{N}$ and $m \in \{1, 2, 3\}$.

An interesting question in this simplest (nontrivial) case of $\ell = 3$ is whether there is a possibility of building the fundamental spinor corepresentation W_1 out of three copies of V_1 or V_2. The decomposition rules permit us to verify easily that the corepresentations $(V_1)^{\otimes 3}$ and $(V_2)^{\otimes 3}$ do not contain W_1 as a subcorepresentation and the same is true for the tensor cube of the irreducible corepresentations $(V_m \otimes W_n)$ if $m \in \{1, 2\}$. They do however contain W_1 as a quotient (sub)corepresentation. It is also worth mentioning that the fundamental spinor W_1 subcorepresentation occurs nevertheless in, e.g., the reducible but not decomposable representation Y_3 and thus also in its third tensor power $(Y_3)^{\otimes 3}$.

4. Braiding

For general q the quasitriangular structure on $U_q(sl(2))$ given by the well-known universal element R [14] in (a suitable completion of) $U_q(sl(2))^{\otimes 2}$, defines a coquasitriangular structure on $A(SL_q(2))$. Its explicit form on the generators reads (cf [17])

$$\mathcal{R}\begin{pmatrix} a\otimes a & a\otimes b & a\otimes c & a\otimes d \\ b\otimes a & b\otimes b & b\otimes c & b\otimes d \\ c\otimes a & c\otimes b & c\otimes c & c\otimes d \\ d\otimes a & d\otimes b & d\otimes c & d\otimes d \end{pmatrix} = \begin{pmatrix} q^{-1/2} & 0 & 0 & q^{1/2} \\ 0 & 0 & q^{-1/2}-q^{3/2} & 0 \\ 0 & 0 & 0 & 0 \\ q^{1/2} & 0 & 0 & q^{-1/2} \end{pmatrix}. \tag{12}$$

This structure provides a highly unusual (nonsymmetric and nondiagonal) braiding of corepresentations ρ and ρ' of $A(SL_q(2))$

$$\Psi(u_i \otimes u'_r) = \sum_{j,s} \mathcal{R}\big(\rho'^{s}_{r} \otimes \rho^{j}_{i}\big) u'_s \otimes u_j. \tag{13}$$

In our situation, $q^\ell = 1$, it is not difficult however to verify that the corepresentations W_n for n odd (i.e. with half-integer spin $n/2$) are fermionic and the corepresentations W_n for n even (i.e. with integer spin $n/2$) are bosonic, i.e. they obey

$$\Psi(w \otimes w') = (-1)^{nn'} w' \otimes w \qquad \text{for} \quad w \in W_n \quad w' \in W_{n'}. \tag{14}$$

Thus the exotic coquasitriangular structure $\mathcal{R}$ passes an important consistency test of the agreement with the usual spin–statistics relation in dimensions $d \geqslant 3$. The braiding of V and W is also quite simple:

$$\Psi(v \otimes w) = (-1)^{mn} w \otimes v \qquad \text{for} \quad v \in V_m \quad w \in W_n. \tag{15}$$

Instead, the braiding of V among themselves is highly exotic (even comparing with the anyonic one). We report it in subsection 4.1 for the case $\ell = 3$. Clearly the tensor products $V_m \otimes W_n$ carry the combined statistics according to the usual hexagon conditions for Ψ (see, e.g., [23]).

4.1. Braiding in the case $\ell = 3$

As for general ℓ, the braiding of the corepresentations W_n with W'_n is exactly the classical one, i.e. the trivial twist except when nn' is odd when it is (-) the twist. Also, W_n have the trivial braiding with V_0 and with V_2 and (-) the twist with V_1. The braiding of V among themselves is as follows.

The braiding of V_1 and V_1 reads

$$\Psi\begin{pmatrix} a\otimes a \\ a\otimes c \\ c\otimes a \\ c\otimes c \end{pmatrix} = \begin{pmatrix} q^{-1/2} & 0 & 0 & 0 \\ 0 & 0 & q^{1/2} & 0 \\ 0 & q^{1/2} & 1+q^{-1/2} & 0 \\ 0 & 0 & 0 & q^{-1/2} \end{pmatrix}\begin{pmatrix} a\otimes a \\ a\otimes c \\ c\otimes a \\ c\otimes c \end{pmatrix}. \tag{16}$$

(Note that Ψ has a nonsimple tensor $a \otimes c - qc \otimes a$ as an eigenvector with eigenvalue 1 and the complex span of $a \otimes a,\ qa \otimes c + c \otimes a,\ c \otimes c$ as an eigenspace with eigenvalue $q^{-1/2}$.) Next, the braiding of V_1 and V_2 reads

$$\Psi \begin{pmatrix} a^2 \otimes a \\ a^2 \otimes c \\ ac \otimes a \\ ac \otimes c \\ c^2 \otimes a \\ c^2 \otimes c \end{pmatrix} = \begin{pmatrix} q^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & q & 0 & 0 \\ 0 & 1 & 0 & q^2-q & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & q & 0 & 1-q & 0 \\ 0 & 0 & 0 & 0 & 0 & q^2 \end{pmatrix} \begin{pmatrix} a \otimes a^2 \\ a \otimes ac \\ a \otimes c^2 \\ c \otimes a^2 \\ c \otimes ac \\ c \otimes c^2 \end{pmatrix}. \tag{17}$$

The opposite braiding of V_2 and V_1 reads

$$\Psi \begin{pmatrix} a \otimes a^2 \\ a \otimes ac \\ a \otimes c^2 \\ c \otimes a^2 \\ c \otimes ac \\ c \otimes c^2 \end{pmatrix} = \begin{pmatrix} q^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & q & 0 \\ 0 & q & q-q^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1-q^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & q^2 \end{pmatrix} \begin{pmatrix} a^2 \otimes a \\ a^2 \otimes c \\ ac \otimes a \\ ac \otimes c \\ c^2 \otimes a \\ c^2 \otimes c \end{pmatrix}. \tag{18}$$

Finally, the braiding of V_2 and V_2 reads

$$\Psi \begin{pmatrix} a^2 \otimes a^2 \\ a^2 \otimes ac \\ a^2 \otimes c^2 \\ ac \otimes a^2 \\ ac \otimes ac \\ ac \otimes c^2 \\ c^2 \otimes a^2 \\ c^2 \otimes ac \\ c^2 \otimes c^2 \end{pmatrix} = \begin{pmatrix} q & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & q & 0 & 0 \\ 0 & 1 & 0 & 1-q & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1-q^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & q^2 & 0 & q-1 & 0 & -(q-1)^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1-q & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q^2 \end{pmatrix} \begin{pmatrix} a^2 \otimes a^2 \\ a^2 \otimes ac \\ a^2 \otimes c^2 \\ ac \otimes a^2 \\ ac \otimes ac \\ ac \otimes c^2 \\ c^2 \otimes a^2 \\ c^2 \otimes ac \\ c^2 \otimes c^2 \end{pmatrix}. \tag{19}$$

The resulting braiding of the irreducible corepresentations $W_n \otimes V_m$ and $W_{n'} \otimes V_{m'}$ can be obtained from the above braidings of V and W using the hexagon conditions for braiding.

5. Final remarks

We remark in connection with point (iii) of proposition 2.1 that Mat(3, $\mathbb{C}$) occurs as a direct summand of Connes' interior algebra $\mathcal{A}$ for the standard model [4]. Also, the algebra $M(3,\mathbb{C}) \oplus M(2,\mathbb{C}) \oplus \mathbb{C}$, close to $\mathcal{A}$, coincides with the semisimple part of the algebra $U_q(sl(2))$ at cubic roots of unity, which was extensively studied (cf [6, 8, 18] and references therein).

We also mention some related works. In [22] the braided group $B(SL_q(2))$ with the braiding induced by the universal R-matrix via the right adjoint corepresentation has been described. In [13] the fractional supersymmetry has been discussed. The interesting papers [24, 25] investigated the spin–statistics relation in the supergroup framework and its bosonization to an (ordinary) Hopf algebra. In [16] the noncommutative cohomology and electromagnetism on $SL_q(2)$ at roots of unity have been studied.

We summarize our work by saying that in noncommutative geometry the spin and the statistics in dimensions $\geqslant$3 look more similar to the case of dimension 2. In fact, to the best of our knowledge, besides the parafermion case, it opens a possibility of unusual statistics in quantum theories in dimension $\geqslant 3$. It is however quite encouraging that the corepresentations W_n of the usual $Spin(3, \mathbb{C})$ maintain their Bose–Fermi statistics in agreement with the usual spin–statistics theorem in (local) relativistic quantum field theory.

There are some open problems regarding the noncommutative spin covers. From the mathematical point of view certainly the question about the cleftness of (7) and the (non)reducibility of (8) to $\mathbb{Z}_2$ should be settled. Also the issue of reality or *-structure and the reductions to $SU(2)$ and $SO(3)$ are very important. Of course the covers of relativistic symmetries (Lorentz and Poincaré) should then be worked out as well. Another task is to employ the quantum covers as structure groups for bundles on (classical or quantum) spaces. These and related topics are currently under investigation.

From the physical point of view, besides the hypothetical relation with quantum symmetries behind the standard model, the main question concerns a possible role of the quantum covers of spin, e.g., in local quantum field theory. Indeed from the discussion above we see that there is an interesting mixing up between the exotic statistics of V and spin. This may be a key test for physical applications. Also the use in physics of the indecomposable corepresentations, their irreducible subcomodules and their quotients require further study.

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